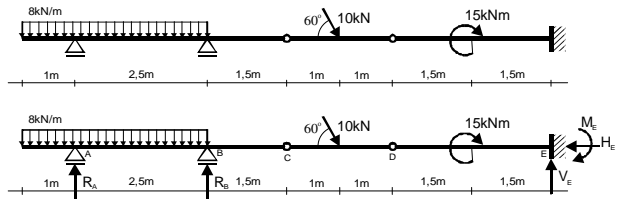


# Przykład – belka przegubowa



# Mechanika teoretyczna

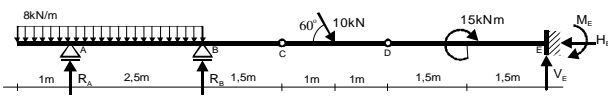
Wykład nr 3

Siły wewnętrzne – przykłady

1

2

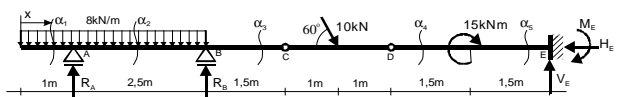
## Reakcje



$$\begin{aligned} \sum X: H_E - 10 \cos 60^\circ &= 0 \\ \sum Y: R_A + R_B + V_E - 8kN/m \cdot 3,5m - 10kN \sin 60^\circ &= 0 \\ \sum M_E: R_A \cdot 9m + R_B \cdot 6,5m + M_E + \\ &+ 15kNm - 10kN \sin 60^\circ \cdot 4m - 8 \cdot 3,5m \cdot \left(\frac{1}{2} \cdot 3,5m + 6,5m\right) = 0 \\ \sum M'_C: R_A \cdot 4m + R_B \cdot 1,5m - 8kN/m \cdot 3,5m \cdot \left(\frac{1}{2} \cdot 3,5m + 1,5m\right) &= 0 \\ \sum M''_D: R_A \cdot 6m + R_B \cdot 3,5m - 8kN/m \cdot 3,5m \cdot \left(\frac{1}{2} \cdot 3,5m + 3,5m\right) - 10kN \sin 60^\circ \cdot 1m &= 0 \end{aligned}$$

3

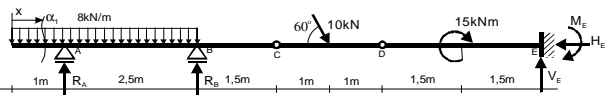
## Przyjęcie przekrojów, przedziały



$$\begin{aligned} R_A &= 17,002kN & a_1 - a_1 & x \in (0; 1m) \\ R_B &= 15,328kN & a_2 - a_2 & x \in (1m; 3,5m) \\ H_E &= 5kN & a_3 - a_3 & x \in (3,5m; 6m) \\ V_E &= 4,330kN & a_4 - a_4 & x \in (6m; 8,5m) \\ M_E &= -2,010kNm & a_5 - a_5 & x \in (8,5m; 10m) \end{aligned}$$

4

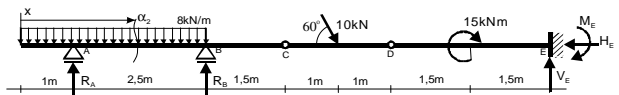
## Przekrój $a_1 - a_1 \quad x \in (0; 1m)$



$$\begin{aligned} N_{a1} &= 0 \\ T_{a1} &= -8kN/m \cdot x & \begin{cases} x=0 & T_{a1} = 0 \\ x=1m & T_{a1} = -8kN \end{cases} \\ M_{a1} &= -8kN/m \cdot x \cdot \frac{x}{2} = -4kN/m \cdot x^2 & \begin{cases} x=0 & M_{a1} = 0 \\ x=1m & M_{a1} = -4kNm \end{cases} \end{aligned}$$

5

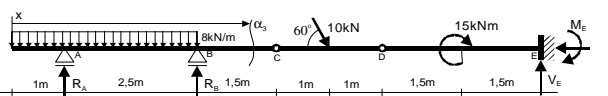
## Przekrój $a_2 - a_2 \quad x \in (1m; 3,5m)$



$$\begin{aligned} N_{a2} &= 0 \\ T_{a2} &= -8kN/m \cdot x + 17,002kN & \begin{cases} x=1m & T_{a2} = 9,002kN \\ x=3,5m & T_{a2} = -10,998kN \end{cases} \\ M_{a2} &= -8kN/m \cdot x \cdot \frac{x}{2} + 17,002kN \cdot (x-1m) = -4kN/m \cdot x^2 + 17,002kN \cdot x - 17,002kNm & \begin{cases} x=1m & M_{a2} = -4kNm \\ x=3,5m & M_{a2} = -6,495kNm \end{cases} \end{aligned}$$

6

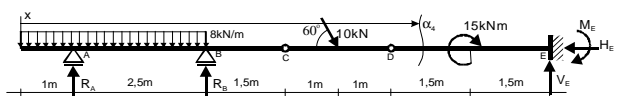
## Przekrój $a_3 - a_3 \quad x \in (3,5m; 6m)$



$$\begin{aligned} N_{a3} &= 0 \\ T_{a3} &= -8kN/m \cdot 3,5m + 17,002kN + 15,328kN = 4,33kN \\ M_{a3} &= -8kN/m \cdot 3,5m \cdot \left(x - \frac{3,5m}{2}\right) + 17,002kN \cdot (x-1m) + 15,328kN \cdot (x-3,5m) = 4,33kN \cdot x - 21,65kNm & \begin{cases} x=3,5m & M_{a3} = -6,495kNm \\ x=5m & M_{a3} = 0 \\ x=6m & M_{a3} = 4,330kNm \end{cases} \end{aligned}$$

7

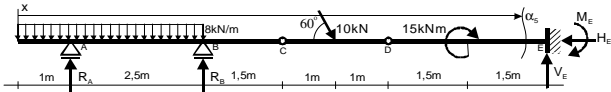
## Przekrój $a_4 - a_4 \quad x \in (6m; 8,5m)$



$$\begin{aligned} N_{a4} &= -10kN \cos 60^\circ = -5kN \\ T_{a4} &= 4,33kN - 10kN \sin 60^\circ = -4,33kN \\ M_{a4} &= 4,33kN \cdot x - 21,65kNm - 10kN \sin 60^\circ \cdot (x-6m) = -4,33kN \cdot x + 30,31kNm & \begin{cases} x=6m & M_{a4} = 4,33m \\ x=7m & M_{a4} = 0 \\ x=8,5m & M_{a4} = -6,495kNm \end{cases} \end{aligned}$$

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## Przekrój $a_5 - a_5$ $x \in \langle 8,5m; 10m \rangle$



$$N_{a_5} = -5kN$$

$$T_{a_5} = -4,33kN$$

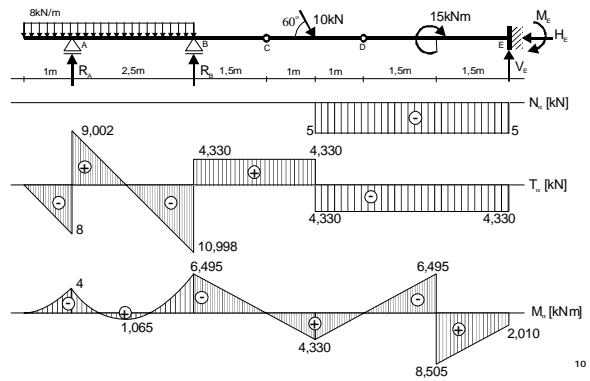
$$M_{a_5} = -4,33kN \cdot x + 30,31kNm + 15kNm =$$

$$= -4,33kN \cdot x + 45,31kNm$$

$$\left. \begin{array}{l} x = 8,5m \quad M_{a_5} = 8,505kNm \\ x = 10m \quad M_{a_5} = 2,01kNm \end{array} \right\}$$

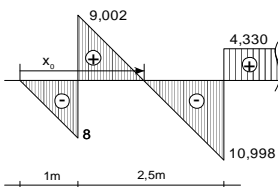
9

## Wykresy



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## Ekstremum



$$\frac{9,002kNm + 10,998kNm}{2,5m} = \frac{9,002kNm}{x_0 - 1m}$$

$T_x$  [kN]

$$T_{a_2} = -8kN/m \cdot x + 17,002kN = 0$$

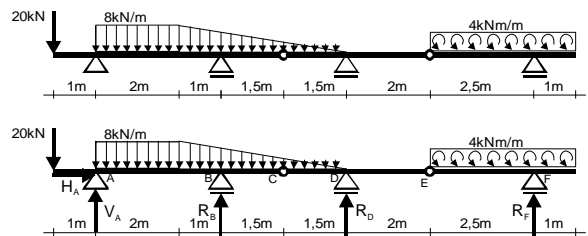
$$x_0 = 2,125m$$

$$M_{a_2} = -4kN/m \cdot x^2 + 17,002kN \cdot x - 17,002kNm$$

$$M_{a_2}(x_0 = 2,125m) = 1,065kNm$$

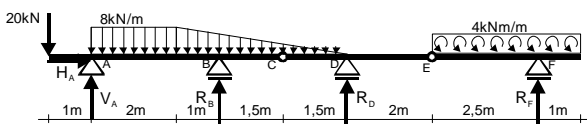
11

## Przykład – belka przegubowa



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## Reakcje



$$\sum X: H_A = 0$$

$$\sum Y: V_A + R_B + R_D + R_F - 20kN - 8kN/m \cdot 2m - \frac{1}{2} \cdot 8kN/m \cdot 4m = 0$$

$$\sum M_A: R_B \cdot 3m + R_D \cdot 6m + R_F \cdot 10,5m + 20kN \cdot 1m - 8kN/m \cdot 2m \cdot 1m +$$

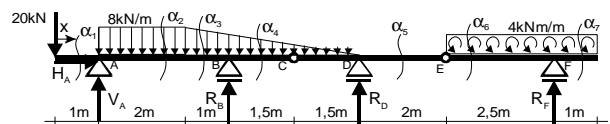
$$- \frac{1}{2} \cdot 8kN/m \cdot 4m \cdot \left(2m + \frac{1}{3} \cdot 4m\right) + 4kNm/m \cdot 3,5m = 0$$

$$\sum M_E^p: R_F \cdot 2,5m + 4kNm/m \cdot 3,5m = 0$$

$$\sum M_C^c: R_D \cdot 1,5m + R_F \cdot 6m - \frac{1}{2} \cdot 3kN/m \cdot 1,5m \cdot \frac{1}{3} \cdot 1,5m + 4kNm/m \cdot 3,5m = 0$$

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## Przyjęcie przekrojów, przedziały



$$H_A = 0$$

$$R_F = 5,6kN$$

$$R_D = 13,817kN$$

$$R_B = 3,744kN$$

$$V_A = 40,039kN$$

$$a_1 - a_1 \quad x \in \langle 0; 1m \rangle$$

$$a_2 - a_2 \quad x \in \langle 1m; 3m \rangle$$

$$a_3 - a_3 \quad x \in \langle 3m; 4m \rangle$$

$$a_4 - a_4 \quad x \in \langle 4m; 7m \rangle$$

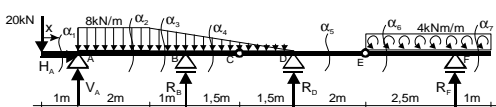
$$a_5 - a_5 \quad x \in \langle 7m; 9m \rangle$$

$$a_6 - a_6 \quad x \in \langle 9m; 11,5m \rangle$$

$$a_7 - a_7 \quad x \in \langle 11,5m; 12,5m \rangle$$

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## Przekrój $a_1 - a_1$ $x \in \langle 0; 1m \rangle$



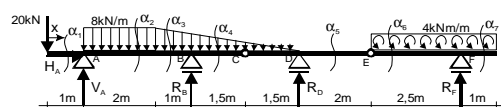
$$N_{a_1} = 0$$

$$T_{a_1} = -20kN$$

$$M_{a_1} = -20kN \cdot x \quad \left| \begin{array}{l} x = 0 \quad M_{a_1} = 0 \\ x = 1m \quad M_{a_1} = -20kNm \end{array} \right.$$

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## Przekrój $a_2 - a_2$ $x \in \langle 1m; 3m \rangle$



$$N_{a_2} = H_A = 0$$

$$T_{a_2} = -20kN + V_A - 8kN/m \cdot (x-1m) = -20kN + 40,039kN - 8kN/m \cdot (x-1m) =$$

$$= 28,039kN - 8kN/m \cdot x$$

$$\left| \begin{array}{l} x = 1m \quad T_{a_2} = 20,039kN \\ x = 3m \quad T_{a_2} = 4,039kN \end{array} \right.$$

$$M_{a_2} = -20kN \cdot x + V_A \cdot (x-1m) - 8kN/m \cdot (x-1m) \cdot \frac{x-1m}{2} =$$

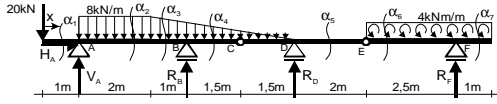
$$= -20kN \cdot x + 40,039kN \cdot (x-1m) - 8kN/m \cdot (x-1m) \cdot \frac{x-1m}{2} =$$

$$= -4kN/m \cdot x^2 + 28,039kN \cdot x - 44,039kNm$$

$$\left| \begin{array}{l} x = 1m \quad M_{a_2} = -20kNm \\ x = 3m \quad M_{a_2} = 4,078kNm \end{array} \right.$$

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## Przekrój $a_3 - a_3$ $x \in \langle 3m; 4m \rangle$ – siły normalne i tnące



$$N_{a_3} = H_A = 0$$

$$q_3(x) = \frac{8kN/m}{7m-x} = \frac{8kN/m}{4m}$$

$$q_3(x) = 2kN/m^2 \cdot (7m-x)$$

$$T_{a_3} = -20kN + V_A - 8kN/m \cdot 2m - q_3(x) \cdot (x-3m) - \frac{1}{2}(8kN/m - q_3(x)) \cdot (x-3m) =$$

$$= -20kN + 40,039kN - 16kN - 2kN/m^2 \cdot (7m-x) \cdot (x-3m) +$$

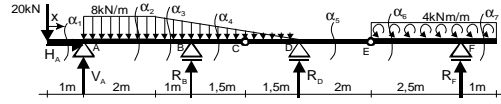
$$- \frac{1}{2}(8kN/m - 2kN/m^2 \cdot (7m-x)) \cdot (x-3m) =$$

$x = 3m$	$T_{a_3} = 4,039kN$
$x = 4m$	$T_{a_3} = -2,961kN$

$$= \frac{1kN}{m^2} x^2 - 14 \frac{kN}{m} x + 37,039kN$$

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## Przekrój $a_3 - a_3$ $x \in \langle 3m; 4m \rangle$ – momenty zginające



$$M_{a_3} = -20kN \cdot x + V_A \cdot (x-1m) - 8kN/m \cdot 2m \cdot (x-2m) +$$

$$- q_3(x) \cdot (x-3m) \cdot \frac{(x-3m)}{2} - \frac{1}{2}(8kN/m - q_3(x)) \cdot (x-3m) \cdot \frac{2}{3}(x-3m) =$$

$$= -20kN \cdot x + 40,039kN \cdot (x-1m) - 16kN \cdot (x-2m) +$$

$$- 2kN/m^2 \cdot (7m-x) \cdot (x-3m) \cdot \frac{(x-3m)}{2} +$$

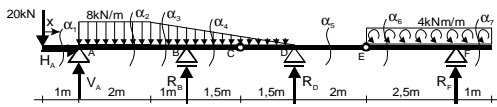
$$- \frac{1}{3}(8kN/m - 2kN/m^2 \cdot (7m-x)) \cdot (x-3m) =$$

$x = 3m$	$M_{a_3} = 4,078kNm$
$x = 4m$	$M_{a_3} = 4,450kNm$

$$= \frac{1}{3}kN/m^2 \cdot x^3 - 7kN/m \cdot x^2 + 37,039kN \cdot x - 53,039kNm$$

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## Przekrój $a_4 - a_4$ $x \in \langle 4m; 7m \rangle$



$$N_{a_4} = H_A = 0$$

$$T_{a_4} = -20kN + V_A - 8kN/m \cdot 2m - q_3(x) \cdot (x-3m) - \frac{1}{2}(8kN/m - q_3(x)) \cdot (x-3m) + R_B =$$

$$= \frac{1kN}{m^2} x^2 - 14 \frac{kN}{m} x + 37,039kN + 3,744kN =$$

$x = 4m$	$T_{a_4} = 0,783kN$
$x = 7m$	$T_{a_4} = -8,217kN$

$$= \frac{1kN}{m^2} x^2 - 14 \frac{kN}{m} x + 40,783kN$$

$$M_{a_4} = -20kN \cdot x + V_A \cdot (x-1m) - 8kN/m \cdot 2m \cdot (x-2m) - q_3(x) \cdot (x-3m) \cdot \frac{(x-3m)}{2} +$$

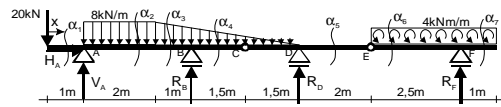
$$- \frac{1}{2}(8kN/m - q_3(x)) \cdot (x-3m) \cdot \frac{2}{3}(x-3m) + R_B \cdot (x-4m) =$$

$x = 4m$	$M_{a_4} = 4,450kNm$
$x = 5,5m$	$M_{a_4} = 0$
$x = 7m$	$M_{a_4} = -11,201kNm$

$$= \frac{1}{3}kN/m^2 \cdot x^3 - 7kN/m \cdot x^2 + 40,783kN \cdot x - 68,015kNm$$

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## Przekrój $a_5 - a_5$ $x \in \langle 7m; 9m \rangle$



$$N_{a_5} = H_A = 0$$

$$T_{a_5} = -20kN + V_A - 8kN/m \cdot 2m - \frac{1}{2}8kN/m \cdot 4m + R_B + R_D =$$

$$= -20kN + 40,039kN - 16kN - 16kN + 3,744kN + 13,817kN =$$

$$= 5,6kN$$

$$M_{a_5} = -20kN \cdot x + V_A \cdot (x-1m) - 8kN/m \cdot 2m \cdot (x-2m) +$$

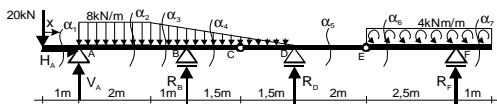
$$- \frac{1}{2}8kN/m \cdot 4m \cdot \left( x - \left( 3m + \frac{1}{3}4m \right) \right) + R_B \cdot (x-4m) + R_D \cdot (x-7m) =$$

$$= 5,6kN \cdot x - 50,401kNm$$

$x = 7m$	$M_{a_5} = -11,201kNm$
$x = 9m$	$M_{a_5} = -0,001kNm \approx 0$

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## Przekrój $a_6 - a_6$ $x \in \langle 9m; 11,5m \rangle$



$$N_{a_6} = H_A = 0$$

$$T_{a_6} = 5,6kN$$

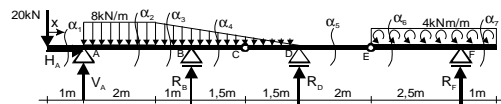
$$M_{a_6} = 5,6kN \cdot x - 50,401kNm - 4kNm/m \cdot (x-9m) =$$

$$= 1,6kN \cdot x - 14,401kNm$$

$x = 9m$	$M_{a_6} = -0,001kNm \approx 0$
$x = 11,5m$	$M_{a_6} = 3,999kNm$

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## Przekrój $a_7 - a_7$ $x \in \langle 11,5m; 12,5m \rangle$



$$N_{a_7} = H_A = 0$$

$$T_{a_7} = 5,6kN - R_F = 0$$

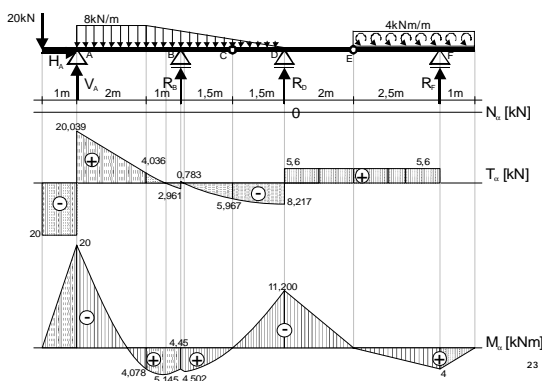
$$M_{a_7} = 1,6kN \cdot x - 14,401kNm - 5,6kN \cdot (x-11,5m) =$$

$$= -4kN \cdot x + 49,999kNm$$

$x = 11,5m$	$M_{a_7} = 3,999kNm$
$x = 12,5m$	$M_{a_7} = -0,001kNm \approx 0$

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## Wykresy



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## Ekstrema

$$T_{a_3} = \frac{1kN}{m^2} x^2 - 14 \frac{kN}{m} x + 37,039kN = 0 \quad x_0 = 3,542m$$

$$M_{a_3}(x_0) = \frac{1}{3}kN/m^2 \cdot x_0^3 - 7kN/m \cdot x_0^2 + 37,039kN \cdot x_0 - 53,039kNm =$$

$$= 5,145kNm$$

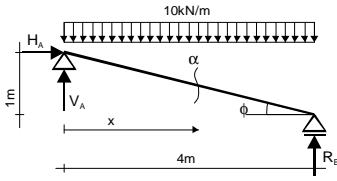
$$T_{a_4} = \frac{1kN}{m^2} x^2 - 14 \frac{kN}{m} x + 40,783kN = 0 \quad x_0 = 4,133m$$

$$M_{a_4}(x_0) = \frac{1}{3}kN/m^2 \cdot x_0^3 - 7kN/m \cdot x_0^2 + 40,783kN \cdot x_0 - 68,015kNm =$$

$$= 4,502kNm$$

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## Obciążenie na pręcie ukośnym – na jednostkę rzutu



$$\sin f = \frac{1m}{\sqrt{(4m)^2 + (1m)^2}} = 0,243$$

$$\cos f = \frac{4m}{\sqrt{(4m)^2 + (1m)^2}} = 0,970$$

$$\sum X : H_A = 0$$

$$H_A = 0$$

$$\sum Y : V_A + R_B - 10 \frac{kN}{m} \cdot 4m = 0$$

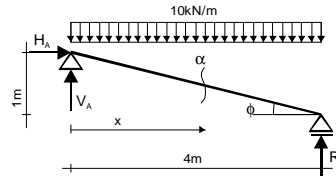
$$V_A = 20kN$$

$$\sum M_A : R_B \cdot 4m - 10 \frac{kN}{m} \cdot 4m \cdot \frac{1}{2} 4m = 0$$

$$R_B = 20kN$$

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## Siły wewnętrzne



$$N_a = V_A \cdot \sin f - q \cdot x \cdot \sin f =$$

$$= 4,851kN - 2,425 \frac{kN}{m} \cdot x$$

$$T_a = V_A \cdot \cos f - q \cdot x \cdot \cos f =$$

$$= 19,403kN - 9,701 \frac{kN}{m} \cdot x$$

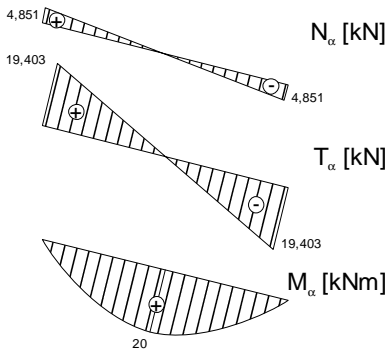
$$M_a = V_A \cdot x - 10 \frac{kN}{m} \cdot x \cdot \frac{x}{2} =$$

$$= -5 \frac{kN}{m} \cdot x^2 + 20kN \cdot x$$

$$\left| \begin{array}{lll} x=0 & N_a = 4,851kN & T_a = 19,403kN & M_a = 0 \\ x=2m & N_a = 0 & T_a = 0 & M_a = 20kNm \\ x=4m & N_a = -4,849kN & T_a = -19,401kN & M_a = 0 \end{array} \right.$$

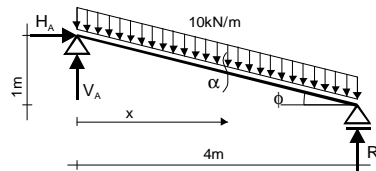
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## Wykresy



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## Obciążenie na pręcie ukośnym – na jednostkę długości pręta



$$\sum X : H_A = 0$$

$$H_A = 0$$

$$\sum Y : V_A + R_B - 10 \frac{kN}{m} \cdot \sqrt{(4m)^2 + (1m)^2} = 0$$

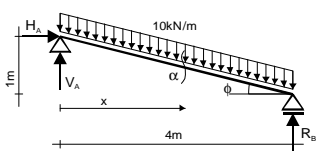
$$V_A = 20,616kN$$

$$\sum M_A : R_B \cdot 4m - 10 \frac{kN}{m} \cdot \sqrt{(4m)^2 + (1m)^2} \cdot \frac{1}{2} 4m = 0$$

$$R_B = 20,616kN$$

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## Siły wewnętrzne



$$N_a = V_A \cdot \sin f - q \cdot \sqrt{x^2 + \left(\frac{x}{4}\right)^2} \cdot \sin f =$$

$$= 5kN - 2,5 \frac{kN}{m} \cdot x$$

$$T_a = V_A \cdot \cos f - q \cdot \sqrt{x^2 + \left(\frac{x}{4}\right)^2} \cdot \cos f =$$

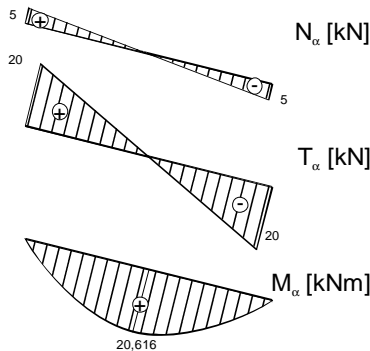
$$= 20kN - 10 \frac{kN}{m} \cdot x$$

$$M_a = V_A \cdot x - 10 \frac{kN}{m} \cdot \sqrt{x^2 + \left(\frac{x}{4}\right)^2} \cdot \frac{x}{2} = -5,154 \frac{kN}{m} \cdot x^2 + 20,616kN \cdot x$$

$$\left| \begin{array}{lll} x=0 & N_a = 5kN & T_a = 20kN & M_a = 0 \\ x=2m & N_a = 0 & T_a = 0 & M_a = 20,616kNm \\ x=4m & N_a = -5 & T_a = -20kN & M_a = 0 \end{array} \right.$$

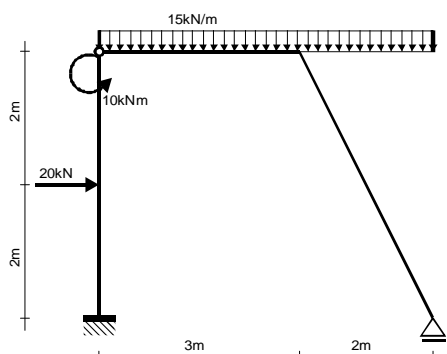
29

## Wykresy



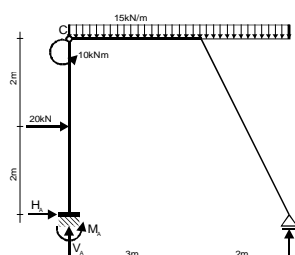
30

## Przykład – rama



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## Reakcje



$$\sum X : H_A + 20kN = 0$$

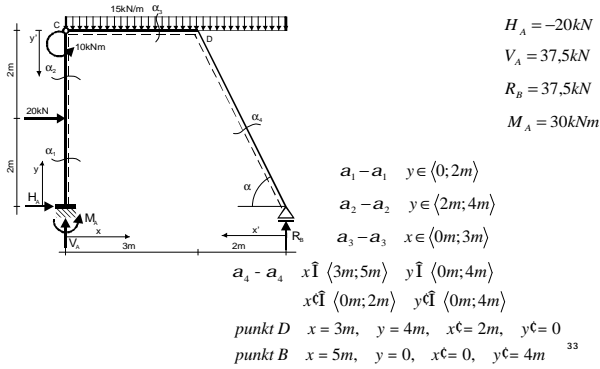
$$\sum Y : V_A + R_B - 15kN \cdot 5m = 0$$

$$\sum M_A : M_A + R_B \cdot 5m + 10kNm - 20kN \cdot 2m - 15kN/m \cdot 5m \cdot \frac{1}{2} 5m = 0$$

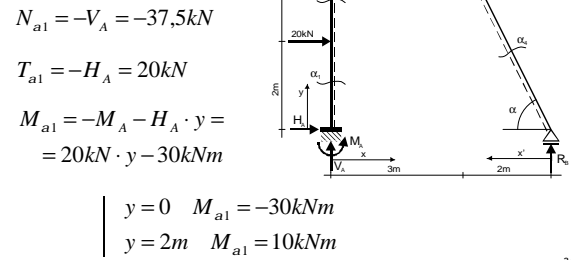
$$\sum M_C : R_B \cdot 5m - 15kN/m \cdot 5m \cdot \frac{1}{2} 5m = 0$$

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## Przyjęcie przekrojów, przedziały

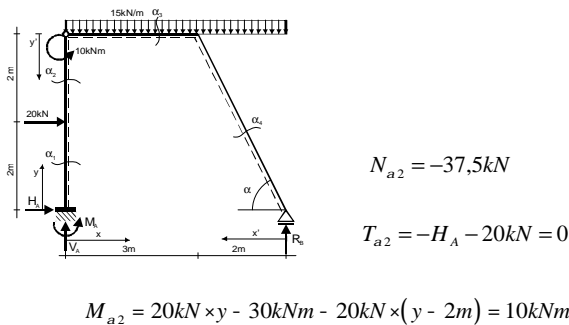


## Przekrój $a_1 - a_1 \quad y \in \langle 0; 2m \rangle$



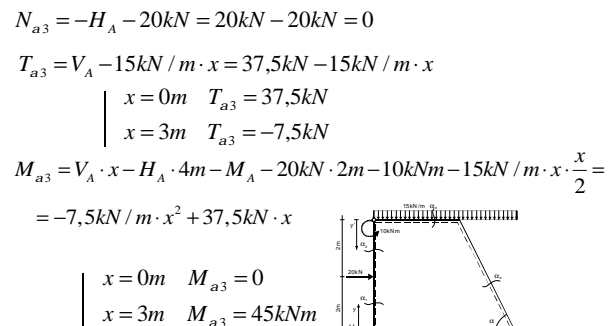
34

## Przekrój $a_2 - a_2 \quad y \in \langle 2m; 4m \rangle$



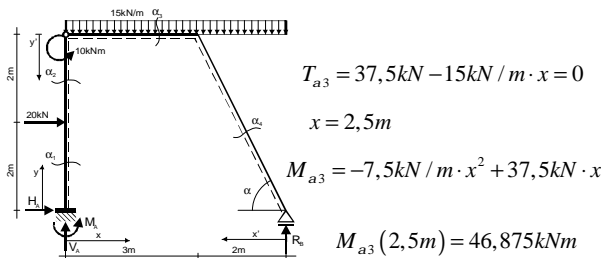
35

## Przekrój $a_3 - a_3 \quad x \in \langle 0; 3m \rangle$



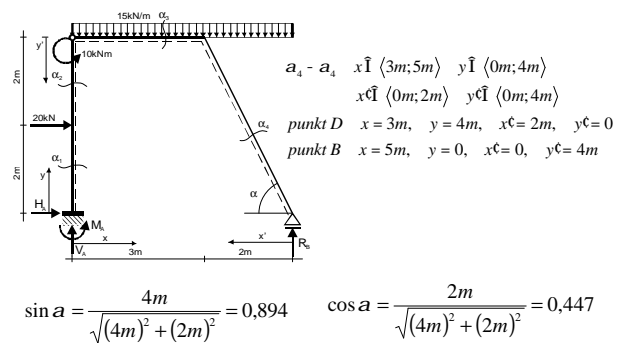
36

## Ekstremum



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## Przekrój $a_4 - a_4$



38

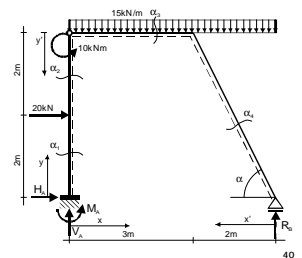
## Przekrój $a_4 - a_4 \quad$ (z lewej strony)

$$\begin{aligned}
 N_{a4} &= -H_A \cos a - 20kN \cos a + V_A \sin a - q \cdot x \sin a = \\
 &= 37,5kN \sin a - 15 \frac{kN}{m} x \sin a = \\
 &= -13,41 \frac{kN}{m} x + 33,525kN \quad \left| \begin{array}{l} x = 3m \quad N_{a4} = -6,705kN \\ x = 5m \quad N_{a4} = -33,525kN \end{array} \right. \\
 T_{a4} &= H_A \sin a + 20kN \sin a + V_A \cos a - q \cdot x \cos a = \\
 &= 37,5kN \cos a - 15 \frac{kN}{m} x \cos a = \\
 &= -6,705 \frac{kN}{m} x + 16,763kN \quad \left| \begin{array}{l} x = 3m \quad N_{a4} = -3,352kN \\ x = 5m \quad N_{a4} = -16,762kN \end{array} \right.
 \end{aligned}$$

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## Przekrój $a_4 - a_4 \quad$ (z lewej strony)

$$\begin{aligned}
 M_{a4} &= V_A \cdot x - H_A \cdot y - 20kN \cdot (y - 2m) - M_A - 15kN/m \cdot x \cdot \frac{x}{2} - 10kNm = \\
 &= 37,5kN \cdot x + 20kN \cdot y - 20kN \cdot y + 40kNm + \\
 &= -30kNm - 7,5 \frac{kN}{m} x^2 - 10kNm = \\
 &= -7,5 \frac{kN}{m} x^2 + 37,5kN \cdot x \\
 & \left| \begin{array}{l} x = 3m \quad M_{a4} = 45kNm \\ x = 5m \quad M_{a4} = 0 \end{array} \right.
 \end{aligned}$$



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## Przekrój $a_4 - a_4$ (z prawej strony)

$$N_{a_4} = -R_B \sin a + q \cdot x' \sin a = -37,5 \text{ kN} \sin a + 15 \frac{\text{kN}}{\text{m}} x' \sin a =$$

$$= 13,41 \frac{\text{kN}}{\text{m}} x' - 33,525 \text{ kN} \quad \left| \begin{array}{l} x' = 0 \quad N_{a_4} = -33,525 \text{ kN} \\ x' = 2 \text{ m} \quad N_{a_4} = -6,705 \text{ kN} \end{array} \right.$$

$$T_{a_4} = -R_B \cos a + q \cdot x' \cos a = -37,5 \text{ kN} \cos a + 15 \frac{\text{kN}}{\text{m}} x' \cos a =$$

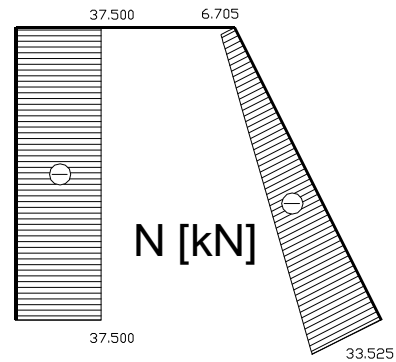
$$= 6,705 \frac{\text{kN}}{\text{m}} x' - 16,763 \text{ kN} \quad \left| \begin{array}{l} x' = 0 \quad T_{a_4} = -16,763 \text{ kN} \\ x' = 2 \text{ m} \quad T_{a_4} = -3,353 \text{ kN} \end{array} \right.$$

$$M_{a_4} = R_B \cdot x' - 15 \text{ kN/m} \cdot x' \cdot \frac{x'}{2} =$$

$$= -7,5 \frac{\text{kN}}{\text{m}} x'^2 + 37,5 \text{ kN} \cdot x' \quad \left| \begin{array}{l} x' = 0 \quad M_{a_4} = 0 \\ x' = 2 \text{ m} \quad M_{a_4} = 45 \text{ kNm} \end{array} \right.$$

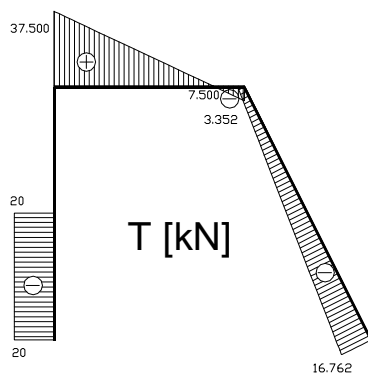
41

## Wykres sił normalnych



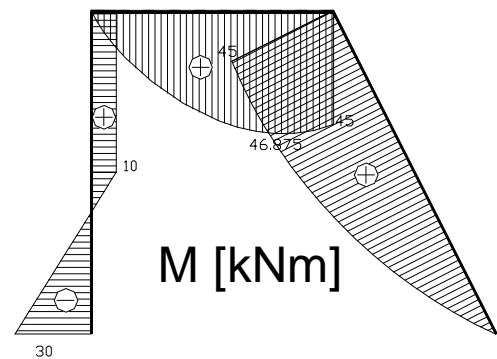
42

## Wykres sił tnących



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## Wykres momentów zginających



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## Kontrola wyników

### n Sprawdzenie równowagi ogólnej

- Suma rzutów sił na nową oś, nierównoległą do wcześniej wykorzystywanych.
- Suma momentów względem dowolnego punktu na płaszczyźnie. Punkt powinien być dobrany tak, aby jak najwięcej reakcji znalazło się w równaniu.

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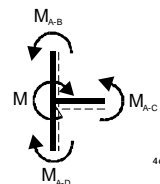
## Kontrola wyników

### n Sprawdzenie równowagi w węzłach:

$$\sum X: T_{A-B} + N_{A-C} - T_{A-D} + P = 0$$

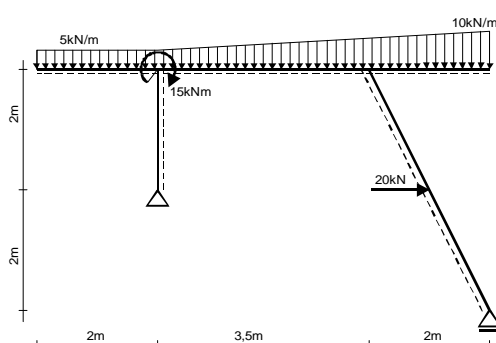
$$\sum Y: N_{A-B} - T_{A-C} - N_{A-D} = 0$$

$$\sum M_A: M_{A-B} + M_{A-C} - M_{A-D} - M = 0$$



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## Przykład – rama



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## Reakcje

$$\sum X: H_A + 20 \text{ kN} = 0$$

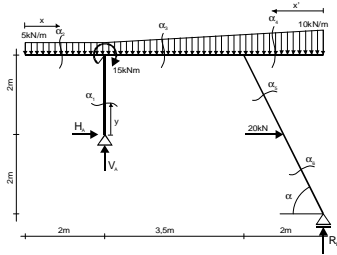
$$\sum Y: V_A + R_B - 5 \frac{\text{kN}}{\text{m}} \cdot 7,5 \text{ m} - \frac{1}{2} \cdot 5 \frac{\text{kN}}{\text{m}} \cdot 5,5 \text{ m} = 0$$

$$\sum M_A: R_B \cdot 5,5 \text{ m} - 15 \text{ kNm} + 5 \frac{\text{kN}}{\text{m}} \cdot 2 \text{ m} \cdot \frac{1}{2} \cdot 2 \text{ m} - 5 \frac{\text{kN}}{\text{m}} \cdot 5,5 \text{ m} \cdot \frac{1}{2} \cdot 5,5 \text{ m} +$$

$$- \frac{1}{2} \cdot 5 \frac{\text{kN}}{\text{m}} \cdot 5,5 \text{ m} \cdot \frac{2}{3} \cdot 5,5 \text{ m} = 0$$

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## Przyjęcie przekrojów, przedziały



$$\begin{aligned} H_A &= -20kN \\ V_A &= 27,424kN \\ R_B &= 23,826kN \end{aligned}$$

$$\begin{aligned} a_1 - a_1 \quad y &\in \langle 0; 2m \rangle \\ a_2 - a_2 \quad x &\in \langle 0; 2m \rangle \\ a_3 - a_3 \quad x &\in \langle 2m; 5,5m \rangle \\ a_4 - a_4 \quad x &\in \langle 5,5m; 7,5m \rangle \\ &\quad x' \in \langle 0; 2m \rangle \\ a_5 - a_5 \quad x &\in \langle 5,5m; 6,5m \rangle \quad y \in \langle 0; 2m \rangle \\ &\quad x' \in \langle 1m; 2m \rangle \\ a_6 - a_6 \quad x &\in \langle 6,5m; 7,5m \rangle \quad y \in \langle -2m; 0 \rangle \\ &\quad x' \in \langle 0; 1m \rangle \end{aligned}$$

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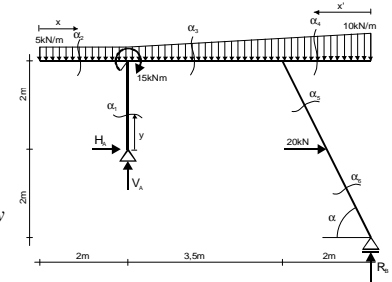
## Przekrój $a_1 - a_1 \quad y \in \langle 0; 2m \rangle$

$$N_{a1} = -V_A = -27,424kN$$

$$T_{a1} = -H_A = 20kN$$

$$M_{a1} = -H_A \cdot y = 20kN \cdot y$$

$$\begin{cases} y = 0 & M_{a1} = 0 \\ y = 2m & M_{a1} = 40kNm \end{cases}$$



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## Przekrój $a_2 - a_2 \quad x \in \langle 0; 2m \rangle$

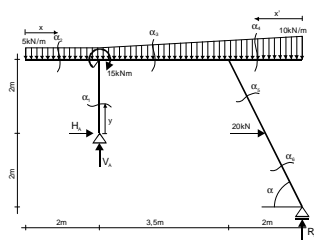
$$N_{a2} = 0$$

$$T_{a2} = -5 \frac{kN}{m} \cdot x$$

$$\begin{cases} x = 0 & T_{a2} = 0 \\ x = 2m & T_{a2} = -10kN \end{cases}$$

$$M_{a2} = -5 \frac{kN}{m} \cdot x \cdot \frac{x}{2} = -2,5 \frac{kN}{m} \cdot x^2$$

$$\begin{cases} x = 0 & M_{a2} = 0 \\ x = 2m & M_{a2} = -10kNm \end{cases}$$



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## Przekrój $a_3 - a_3 \quad x \in \langle 2m; 5,5m \rangle$

$$N_{a3} = -H_A = 20kN$$

$$\frac{5 \frac{kN}{m}}{q'} = \frac{5,5m}{x-2m} \quad q' = 0,909 \frac{kN}{m^2} (x-2m) = 0,909 \frac{kN}{m^2} x - 1,818 \frac{kN}{m}$$

$$T_{a3} = V_A - 5 \frac{kN}{m} \cdot x - \frac{1}{2} q' (x-2m) =$$

$$= 27,424kN - 5 \frac{kN}{m} \cdot x - \frac{1}{2} \left( 0,909 \frac{kN}{m^2} x - 1,818 \frac{kN}{m} \right) (x-2m) =$$

$$= -0,455 \frac{kN}{m^2} x^2 - 3,182 \frac{kN}{m} x + 25,606kN$$

$$\begin{cases} x = 2m & T_{a3} = 17,422kN \\ x = 5,5m & T_{a3} = -5,659kN \end{cases}$$

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## Przekrój $a_3 - a_3 \quad x \in \langle 2m; 5,5m \rangle$

$$M_{a3} = V_A \cdot (x-2m) - H_A \cdot 2m - 5 \frac{kN}{m} \cdot x \cdot \frac{x}{2} +$$

$$- \frac{1}{2} q' \cdot (x-2m) \cdot \frac{1}{3} (x-2m) + 15kNm =$$

$$= 27,424kN \cdot (x-2m) + 20kN \cdot 2m - 2,5 \frac{kN}{m} \cdot x^2 +$$

$$- \frac{1}{6} \left( 0,909 \frac{kN}{m^2} x - 1,818 \frac{kN}{m} \right) \cdot (x-2m)^2 + 15kNm =$$

$$= -0,156 \frac{kN}{m^2} x^3 - 1,591 \frac{kN}{m} x^2 + 25,606kN \cdot x + 1,364kNm$$

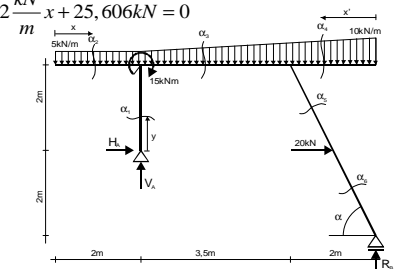
$$\begin{cases} x = 2m & M_{a3} = 44,964kNm \\ x = 5,5m & M_{a3} = 68,115kNm \end{cases}$$

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## Ekstremum

$$T_{a3} = -0,455 \frac{kN}{m^2} x^2 - 3,182 \frac{kN}{m} x + 25,606kN = 0$$

$$x = 4,78m$$



$$M_{a3} = -0,156 \frac{kN}{m^2} x^3 - 1,591 \frac{kN}{m} x^2 + 25,606kN \cdot x + 1,364kNm$$

$$M_{a3}(4,78m) = 70,371kNm$$

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## Przekrój $a_4 - a_4 \quad x \in \langle 5,5m; 7,5m \rangle$

$$N_{a4} = 0$$

$$T_{a4} = 5 \frac{kN}{m} \cdot (7,5m - x) + q'(7,5m - x) + \frac{1}{2} \left( 5 \frac{kN}{m} - q' \right) (7,5m - x) =$$

$$= 37,5kN - 5 \frac{kN}{m} \cdot x - \left( 0,909 \frac{kN}{m^2} x - 1,818 \frac{kN}{m} \right) (7,5m - x) +$$

$$+ \frac{1}{2} \left( 5 \frac{kN}{m} - 0,909 \frac{kN}{m^2} x + 1,808 \frac{kN}{m} \right) (7,5m - x) =$$

$$= -0,454 \frac{kN}{m^2} x^2 - 3,182 \frac{kN}{m} x + 49,433kN$$

$$\begin{cases} x = 5,5m & T_{a4} = 18,199kN \\ x = 7,5m & T_{a4} = 0,031kN \approx 0 \end{cases}$$

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## Przekrój $a_4 - a_4 \quad x \in \langle 5,5m; 7,5m \rangle$

$$M_{a4} = -5 \frac{kN}{m} \cdot (7,5m - x) \cdot \frac{1}{2} (7,5m - x) - q' (7,5m - x) \cdot \frac{1}{2} (7,5m - x) +$$

$$- \frac{1}{2} \left( 5 \frac{kN}{m} - q' \right) (7,5m - x) \cdot \frac{2}{3} (7,5m - x) =$$

$$= -5 \frac{kN}{m} \cdot \frac{1}{2} (7,5m - x)^2 - \left( 0,909 \frac{kN}{m^2} - 1,818 \frac{kN}{m} \right) \frac{1}{2} (7,5m - x)^2 +$$

$$- \frac{1}{2} \left( 5 \frac{kN}{m} - \left( 0,909 \frac{kN}{m^2} - 1,818 \frac{kN}{m} \right) \right) \frac{2}{3} (7,5m - x)^2 =$$

$$= -0,152 \frac{kN}{m^2} x^3 - 1,591 \frac{kN}{m} x^2 + 49,433kN \cdot x - 217,331kNm$$

$$\begin{cases} x = 5,5m & M_{a4} = -18,886kNm \\ x = 7,5m & M_{a4} = -0,202kNm \approx 0 \end{cases}$$

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## Przekrój $a_4 - a_4$

n Zwiększenie dokładności przy  $x$  w najwyższej potędze (większa liczba cyfr znaczących):

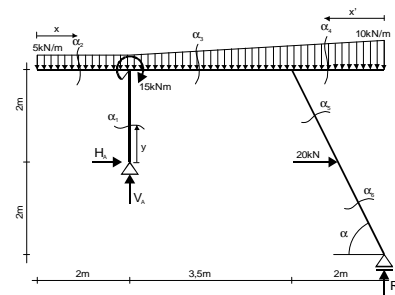
$$T_{a_4} = -0,4545 \frac{kN}{m^2} x^2 - 3,182 \frac{kN}{m} x + 49,433 kN$$

$$M_{a_4} = -0,1515 \frac{kN}{m^2} x^3 - 1,591 \frac{kN}{m} x^2 + 49,433 kN \cdot x - 217,331 kNm$$

$$\begin{cases} x = 5,5m & T_{a_4} = 18,183 kN & M_{a_4} = -18,783 kNm \\ x = 7,5m & T_{a_4} = 0,002 kN \approx 0 & M_{a_4} = 0,009 kNm \end{cases}$$

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## Przekrój $a_6 - a_6$ $x' \in \langle 0; 1m \rangle$



$$\sin a = \frac{4m}{\sqrt{(4m)^2 + (2m)^2}} = 0,894 \quad \cos a = \frac{2m}{\sqrt{(4m)^2 + (2m)^2}} = 0,447$$

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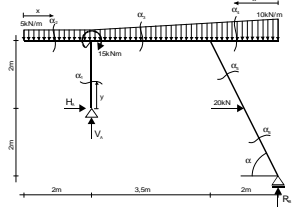
## Przekrój $a_6 - a_6$ $x' \in \langle 0; 1m \rangle$

$$N_{a_6} = -R_B \cdot \sin a = -23,826 kN \cdot 0,894 = -21,300 kN$$

$$T_{a_6} = -R_B \cdot \cos a = -23,826 kN \cdot 0,447 = -10,650 kN$$

$$M_{a_6} = R_B \cdot x' = 23,826 kN \cdot x'$$

$$\begin{cases} x' = 0 & M_{a_6} = 0 \\ x' = 1m & M_{a_6} = 23,826 kNm \end{cases}$$



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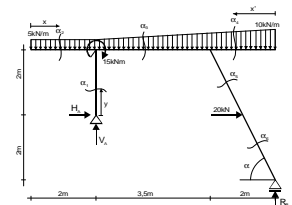
## Przekrój $a_5 - a_5$ $x' \in \langle 1m; 2m \rangle$

$$\begin{aligned} N_{a_5} &= -R_B \cdot \sin a + 20 \cos a = \\ &= -23,826 kN \cdot 0,894 + 20 kN \cdot 0,447 = -12,360 kN \end{aligned}$$

$$\begin{aligned} T_{a_5} &= -R_B \cdot \cos a - 20 \sin a = \\ &= -23,826 kN \cdot 0,447 - 20 kN \cdot 0,894 = -28,530 kN \end{aligned}$$

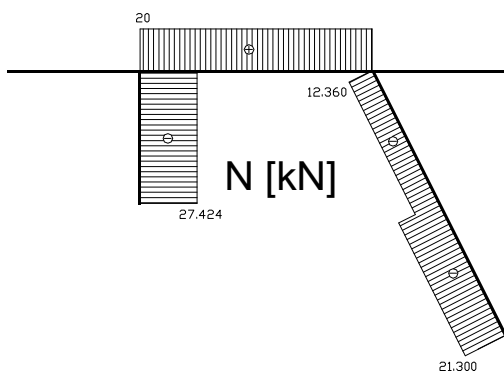
$$\begin{aligned} M_{a_5} &= R_B \cdot x' + 20 \cdot y = \\ &= 23,826 kN \cdot x' + 20 kN \cdot y \end{aligned}$$

$$\begin{cases} x' = 1m & y = 0 & M_{a_5} = 23,826 kNm \\ x' = 2m & y = 2m & M_{a_5} = 87,652 kNm \end{cases}$$



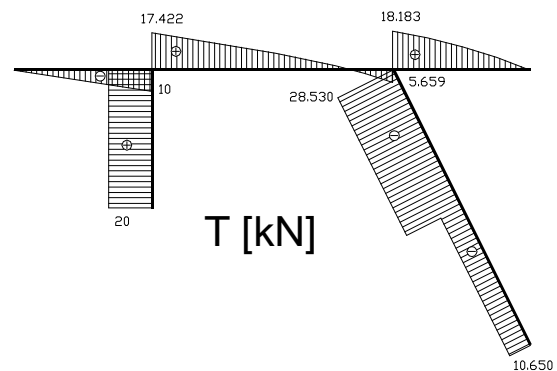
60

## Wykres sił normalnych



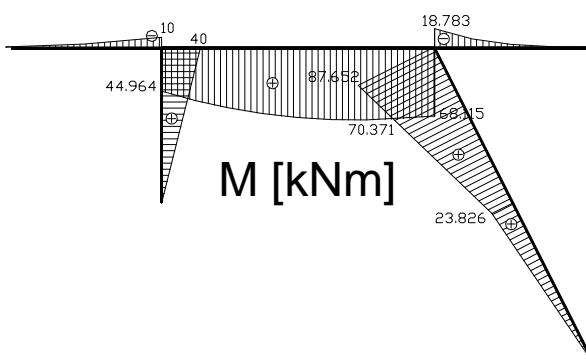
61

## Wykres sił tnących



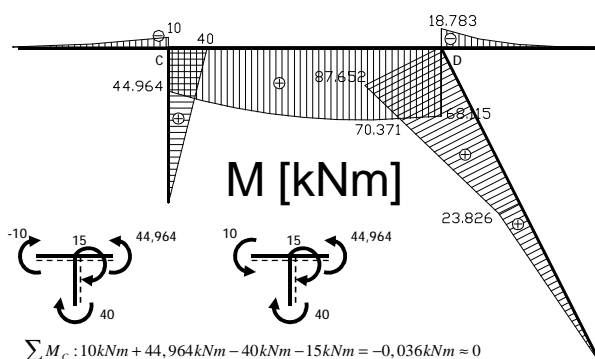
62

## Wykres momentów zginających



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## Sprawdzenie w węźle C – momenty węzłowe

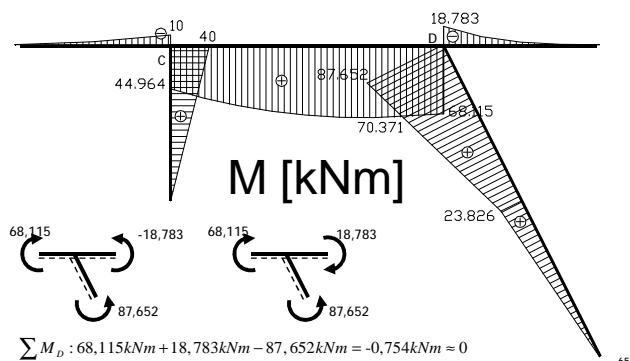


$$\sum M_C : 10 kNm + 44,964 kNm - 40 kNm - 15 kNm = -0,036 kNm \approx 0$$

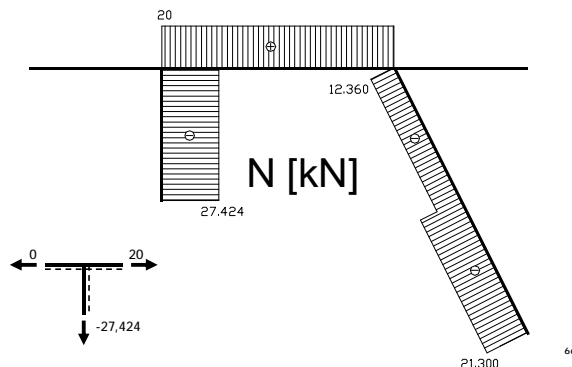
64



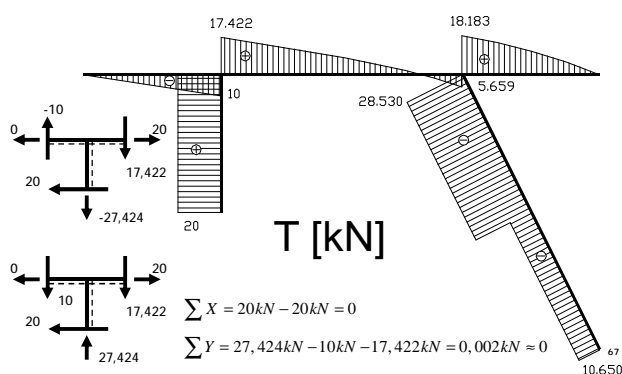
## Sprawdzenie w węźle D – momenty węzłowe



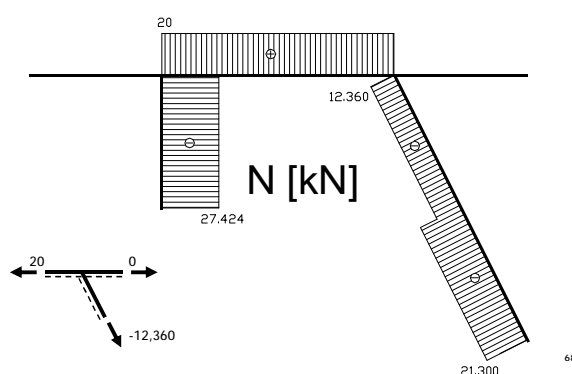
## Węzeł C – normalne



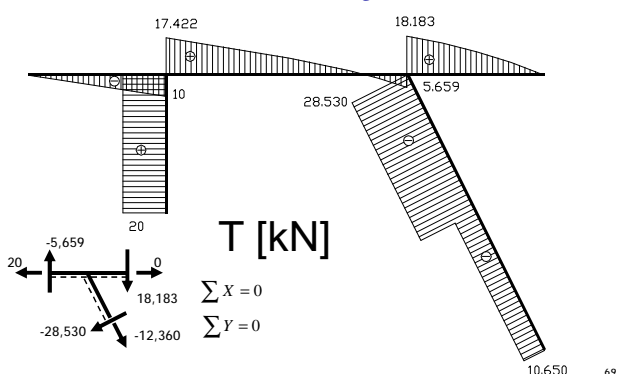
## Sprawdzenie w węźle C – normalne i tnące



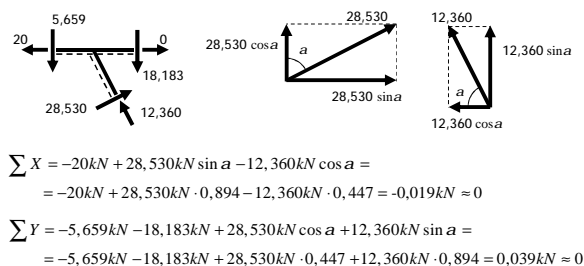
## Węzeł D – normalne



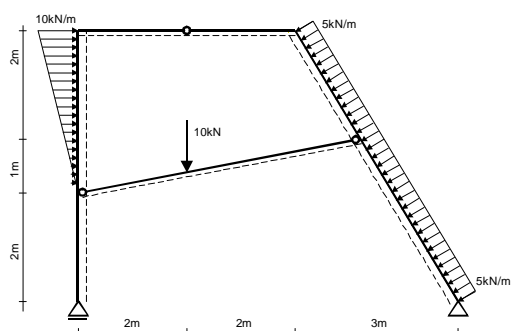
## Sprawdzenie w węźle D – normalne i tnące



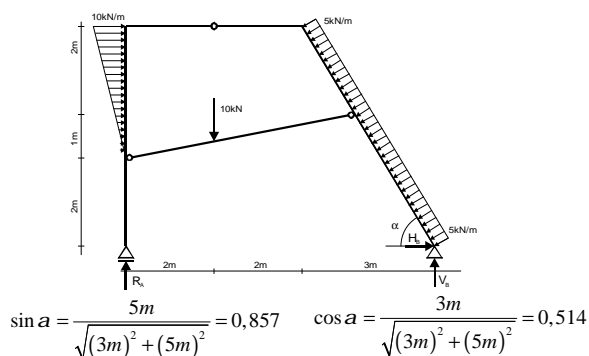
## Sprawdzenie w węźle D – normalne i tnące



## Przykład – rama ze ściągiem



## Reakcje podporowe



## Reakcje podporowe

$$\sum X: H_B + \frac{1}{2} 10 \frac{kN}{m} \cdot 3m - 5 \frac{kN}{m} \cdot \sqrt{(3m)^2 + (5m)^2} \sin \alpha = 0$$

$$\sum Y: R_A + V_B - 10kN - 5 \frac{kN}{m} \cdot \sqrt{(3m)^2 + (5m)^2} \cos \alpha = 0$$

$$\sum M_B: R_A \cdot 7m + \frac{1}{2} 10 \frac{kN}{m} \cdot 3m \cdot \left( 2m + \frac{2}{3} 3m \right) - 10kN \cdot 5m +$$

$$- 5 \frac{kN}{m} \cdot \sqrt{(3m)^2 + (5m)^2} \cdot \frac{1}{2} \sqrt{(3m)^2 + (5m)^2} = 0$$

$$H_B = 9,986kN$$

$$R_A = 10,714kN$$

$$V_B = 14,272kN$$

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## Reakcje w przegubach ściągu

$$\sum X: H_D - H_E = 0 \quad \sum Y: V_D + V_E - 10kN = 0$$

$$H_D = 2,247kN$$

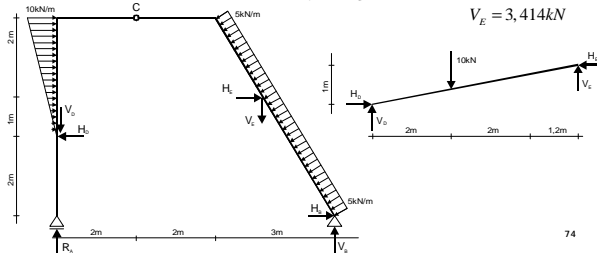
$$\sum M_D: V_E \cdot 5,2m + H_E \cdot 1m - 10kN \cdot 2m = 0$$

$$H_E = 2,247kN$$

$$\sum M'_C: R_A \cdot 2m - V_D \cdot 2m + H_D \cdot 3m - \frac{1}{2} 10 \frac{kN}{m} \cdot 3m \cdot \frac{1}{3} 3m = 0$$

$$V_D = 6,586kN$$

$$V_E = 3,414kN$$



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## Przyjęcie przekrojów, przedziały

$$a_1 - a_1 \quad y \in \langle 0; 2m \rangle$$

$$a_2 - a_2 \quad y \in \langle 2m; 5m \rangle$$

$$a_3 - a_3 \quad x \in \langle 0; 4m \rangle$$

$$a_4 - a_4 \quad x \in \langle 4m; 5,2m \rangle \quad y \in \langle 3m; 5m \rangle$$

$$x' \in \langle 1,8m; 3m \rangle$$

$$a_5 - a_5 \quad x \in \langle 5,2m; 7m \rangle \quad y \in \langle 0; 3m \rangle$$

$$x' \in \langle 0; 1,8m \rangle$$

$$a_6 - a_6 \quad x \in \langle 0; 2m \rangle \quad y \in \langle 2m; 2,6m \rangle$$

$$a_7 - a_7 \quad x \in \langle 2m; 5,2m \rangle \quad y \in \langle 2,6m; 3m \rangle$$

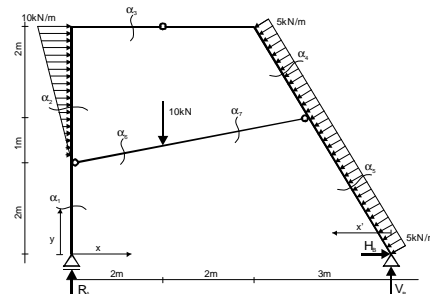
75

## Przekrój $a_1 - a_1 \quad y \in \langle 0; 2m \rangle$

$$N_{a_1} = -R_A = -10,714kN$$

$$T_{a_1} = 0$$

$$M_{a_1} = 0$$



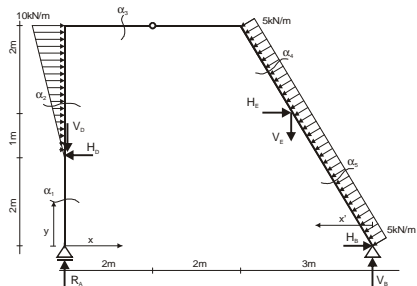
76

## Przekrój $a_2 - a_2 \quad y \in \langle 2m; 5m \rangle$

$$N_{a_2} = -R_A + V_D = -10,714kN + 6,586kN = -4,128kN$$

$$\frac{10 \frac{kN}{m}}{q'} = \frac{3m}{y-2m}$$

$$q' = 3,333 \frac{kN}{m^2} (y-2m)$$



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## Przekrój $a_2 - a_2 \quad y \in \langle 2m; 5m \rangle$

$$T_{a_2} = H_D - \frac{1}{2} q' \cdot (y-2m)$$

$$= 2,247kN - \frac{1}{2} 3,333 \frac{kN}{m^2} \cdot (y-2m)^2 =$$

$$= -1,667 \frac{kN}{m^2} y^2 + 6,666kN \cdot y - 4,419kN$$

$$\left| \begin{array}{l} y=2m \quad T_{a_2} = 2,245kN \\ y=5m \quad T_{a_2} = -12,764kN \end{array} \right.$$

$$M_{a_2} = H_D \cdot (y-2m) - \frac{1}{2} q' \cdot (y-2m) \cdot \frac{1}{3} (y-2m) =$$

$$= 2,247kN \cdot (y-2m) - \frac{1}{6} 3,333 \frac{kN}{m^2} (y-2m)^3 =$$

$$= -0,5555 \frac{kN}{m^2} \cdot y^3 + 3,333 \frac{kN}{m} \cdot y^2 - 4,419kN \cdot y - 0,050kNm$$

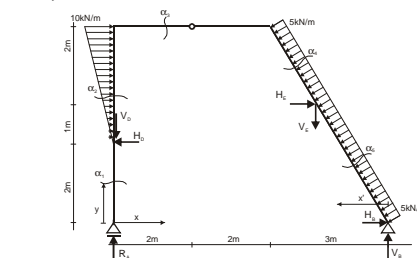
$$\left| \begin{array}{l} y=2m \quad M_{a_2} = 0 \\ y=5m \quad M_{a_2} = -8,258kNm \end{array} \right.$$

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## Ekstremum

$$T_{a_2} = -1,667 \frac{kN}{m^2} y^2 + 6,666kN \cdot y - 4,419kN = 0$$

$$y = 3,16m$$



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## Przekrój $a_3 - a_3 \quad x \in \langle 0; 4m \rangle$

$$N_{a_3} = H_D - \frac{1}{2} 10 \frac{kN}{m} \cdot 3m =$$

$$= 2,247kN - 15kN = -12,753kN$$

$$T_{a_3} = R_A - V_D = 10,714kN - 6,586kN = 4,128kN$$

$$M_{a_3} = R_A \cdot x - V_D \cdot x + H_D \cdot 3m - \frac{1}{2} 10 \frac{kN}{m} \cdot 3m \cdot \frac{1}{3} 3m =$$

$$= 10,714kN \cdot x - 6,586kN \cdot x + 2,247kN \cdot 3m - 15kNm =$$

$$= 4,128kN \cdot x - 8,259kNm$$

$$\left| \begin{array}{l} x=0m \quad M_{a_3} = -8,259kNm \\ x=2m \quad M_{a_3} = -0,003kNm \approx 0 \\ x=4m \quad M_{a_3} = 8,253kNm \end{array} \right.$$

$$M_{a_2} = -0,5555 \frac{kN}{m^2} \cdot y^3 + 3,333 \frac{kN}{m} \cdot y^2 - 4,419kN \cdot y - 0,050kNm$$

$$M_{a_2}(3,16m) = 1,739kNm$$

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## Przekrój $a_5 - a_5$ $x' \in \langle 0m; 1,8m \rangle$

$$N_{a_5} = H_B \cos a - V_B \sin a = 9,986kN \cdot 0,514 - 14,272kN \cdot 0,857 = -7,098kN$$

$$T_{a_5} = -H_B \sin a - V_B \cos a + 5 \frac{kN}{m} \sqrt{x'^2 + y^2} = -9,986kN \cdot 0,857 - 14,272kN \cdot 0,514 + 5 \frac{kN}{m} \sqrt{x'^2 + y^2} = -15,894kN + 5 \frac{kN}{m} \sqrt{x'^2 + y^2} = -15,894kN + 9,720 \frac{kN}{m} x' \quad \left| \begin{array}{l} x' = 0m \quad T_{a_5} = -15,894kN \\ x' = 1,8m \quad T_{a_5} = 1,602kN \end{array} \right.$$

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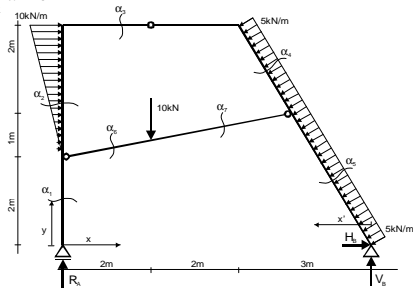
## Przekrój $a_5 - a_5$ $x' \in \langle 0m; 1,8m \rangle$

$$M_{a_5} = V_B \cdot x' + H_B \cdot y - 5 \frac{kN}{m} \sqrt{x'^2 + y^2} \cdot \frac{1}{2} \sqrt{x'^2 + y^2} = 14,272kN \cdot x' + 9,986kN \cdot y - 2,5 \frac{kN}{m} (x'^2 + y^2) = 14,272kN \cdot x' + 9,986kN \cdot 1,667x' - 2,5 \frac{kN}{m} (x'^2 + (1,667x')^2) = -9,447 \frac{kN}{m} x'^2 + 30,919kN \cdot x' \quad \left| \begin{array}{l} x' = 0 \quad M_{a_5} = 0 \\ x' = 1,8m \quad M_{a_5} = 25,046kNm \end{array} \right.$$

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## Ekstremum

$$T_{a_5} = -15,894kN + 9,720 \frac{kN}{m} x' = 0 \quad x' = 1,635m$$



$$M_{a_5} = -9,447 \frac{kN}{m} x'^2 + 30,919kN \cdot x' \quad M_{a_5}(1,635m) = 25,299kNm$$

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## Przekrój $a_4 - a_4$ $x' \in \langle 1,8m; 3m \rangle$

$$N_{a_4} = -7,098kN + H_E \cos a + V_E \sin a = -7,098kN + 2,247kN \cdot 0,514 + 3,414kN \cdot 0,857 = -3,008kN$$

$$T_{a_4} = -15,894kN + 9,720 \frac{kN}{m} x' - H_E \sin a + V_E \cos a = -15,894kN + 9,720 \frac{kN}{m} x' - 2,247kN \cdot 0,857 + 3,414kN \cdot 0,514 = 9,720 \frac{kN}{m} x' - 16,065kN \quad \left| \begin{array}{l} x' = 0 \quad T_{a_4} = 1,431kN \\ x' = 1,8m \quad T_{a_4} = 13,095kN \end{array} \right.$$

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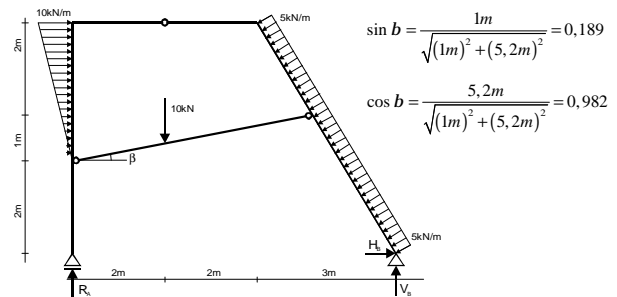
## Przekrój $a_4 - a_4$ $x' \in \langle 1,8m; 3m \rangle$

$$M_{a_4} = -9,447 \frac{kN}{m} x'^2 + 30,919kN \cdot x' + H_E \cdot (y - 3m) - V_E \cdot (x' - 1,8m) = -9,447 \frac{kN}{m} x'^2 + 30,919kN \cdot x' + 2,247kN (1,667x' - 3m) + -3,414kN \cdot (x' - 1,8m) = -9,447 \frac{kN}{m} x'^2 + 31,251kN \cdot x' - 0,596kNm$$

$$\left| \begin{array}{l} x' = 1,8m \quad M_{a_4} = 25,048kNm \\ x' = 3m \quad M_{a_4} = 8,134kNm \end{array} \right.$$

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## Przekrój $a_6 - a_6$ $x \in \langle 0; 2m \rangle$



$$\sin b = \frac{1m}{\sqrt{(1m)^2 + (5,2m)^2}} = 0,189$$

$$\cos b = \frac{5,2m}{\sqrt{(1m)^2 + (5,2m)^2}} = 0,982$$

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## Przekrój $a_6 - a_6$ $x \in \langle 0; 2m \rangle$

$$N_{a_6} = -H_D \cos b - V_D \sin b = -2,247kN \cdot 0,982 - 6,586kN \cdot 0,189 = -3,451kN$$

$$T_{a_6} = -H_D \sin b + V_D \cos b = -2,247kN \cdot 0,189 + 6,586kN \cdot 0,982 = 6,043kN$$

$$M_{a_6} = -H_D \cdot (y - 2m) + V_D \cdot x = -2,247kN \cdot \left( \frac{x}{5,2} + 2m - 2m \right) + 6,586kN \cdot x = -0,432x + 6,586x = 6,154x \quad \left| \begin{array}{l} x = 0m \quad M_{a_6} = 0 \\ x = 2m \quad M_{a_6} = 12,308kNm \end{array} \right.$$

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## Przekrój $a_7 - a_7$ $x \in \langle 2m; 5,2m \rangle$

$$N_{a_7} = -3,451kN + 10kN \sin b = -3,451kN + 10kN \cdot 0,189 = -1,561kN$$

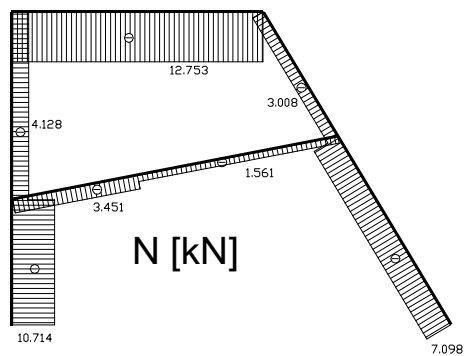
$$T_{a_7} = 6,043kN - 10kN \cos b = 6,043kN - 10kN \cdot 0,982 = -3,777kN$$

$$M_{a_7} = 6,154x - 10kN \cdot (x - 2m) = -3,846x + 20kNm$$

$$\left| \begin{array}{l} x = 2m \quad M_{a_7} = 12,308kNm \\ x = 5,2m \quad M_{a_7} = 0,001kNm \approx 0 \end{array} \right.$$

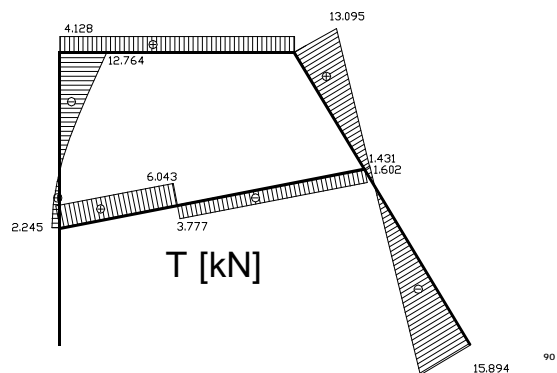
88

## Wykres sił normalnych



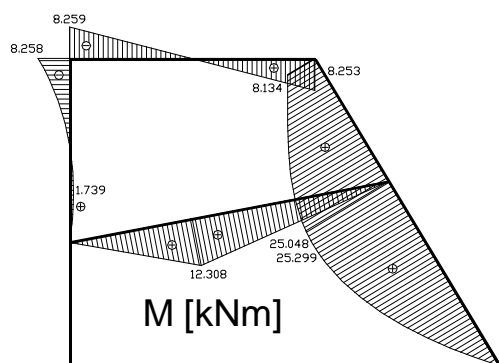
89

## Wykres sił tnących



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## Wykres momentów zginających



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